QUANTITATIVE STABILITY ESTIMATE FOR AN OPTIMIZATION PROBLEM UNDER CONSTRAINTS

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ABSTRACT. A class of functionals maximized by characteristic functions of balls is identified by a mass transportation argument.

A variational approach to the study of standing waves for the nonlinear Schrödinger equation leads to the minimization of functionals like

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\, dx - \int_{\mathbb{R}^n} F(|x|, u(x))\, dx\,, \tag{1}$$

over all $u \in H^1(\mathbb{R}^n)$, $u \geq 0$, such that $\int_{\mathbb{R}^n} u^2 = 1$, see [3] (here $n \in \mathbb{N}, n \geq 1$). The function F describes the index of refraction of the media in which the wave propagates. A typical example is

$$F(r, s) = p(r)s^2 + q(r)s^d\,, \quad 2 < d < 2 + \frac{4}{n}\,,$$

where p and q are positive decreasing functions, and the constraint on d has to be assumed so to avoid non-existence issues due to unbalanced scalings. The two terms of the energy (1) are in competition. Indeed, if we try to maximize $\int_{\mathbb{R}^n} F(|x|, u(x))dx$ under the additional constraint that $u \leq a$, then the unique maximizer is given by the function $a\,1_{rB}$, having infinite Dirichlet integral (here B is the Euclidean unit ball and $r > 0$ is such that $\int_{\mathbb{R}^n}(a\,1_{rB})^2 = 1$).

In this note we identify a simple sufficient condition on an integrand F ensuring that $\int_{\mathbb{R}^n} F(|x|, u(x))dx$ presents this behavior. More precisely, we are going to consider integrands $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ (here $\mathbb{R}^+ := [0, \infty)$), such that

(H1) for every $s \in \mathbb{R}^+$, $F(\cdot, s)$ is decreasing; for a.e. $r \in \mathbb{R}^+$, $F(r, \cdot)$ is continuous on $\mathbb{R}^+$;

(H2) there exist $\alpha \in L^1(\mathbb{R}^+, r^{n-1}dr)$ and a locally bounded function $\beta : \mathbb{R} \to \mathbb{R}^+$ such that, for a.e. $r \in \mathbb{R}^+$ and every $s \in \mathbb{R}^+$, $F(r, s) \leq \alpha(r)\beta(s)$.

Given $a > 0$ and $p \geq 1$, we consider the convex subset of $L^p(\mathbb{R}^n)$

$$X := \left\{ u \in L^p(\mathbb{R}^n) : 0 \leq u \leq a\,, \int_{\mathbb{R}^n} u^p \leq 1 \right\}\,,$$

and define a functional $\mathcal{F}$ on X by setting

$$\mathcal{F}(u) = \int_{\mathbb{R}^n} F(|x|, u(x))dx\,, \quad \forall u \in X\,.$$

Note that, thanks to (H1) and (H2), $x \in \mathbb{R}^n \mapsto F(|x|, u(x))$ is measurable and $\mathcal{F}(u) \in \mathbb{R}^+$ for every $u \in X$. We are going to prove the following theorem:

Theorem 1. *Let $a > 0$, $p \geq 1$, and let F be such that (H1) and (H2) hold true. Assume that there exists $t > 0$ such that the ball $E = \{x \in \mathbb{R}^n : F(|x|, a) > t\}$*

satisfies $a^p|E| = 1$, and that, for a.e. $r \in \mathbb{R}^+$ and for every $\lambda \in [0,1]$,

$$F(r, \lambda a) \le \lambda^p F(r,a)\,. \tag{2}$$

Then the function $w = a\,1_E$ is a maximum of $\mathcal{F}$ on X. Moreover, if $F(\cdot, a)$ is strictly decreasing, then $w = a\,1_E$ is the unique maximizer of $\mathcal{F}$ on X.

The proof of Theorem 1 is based on a basic result in mass transportation theory, namely the Brenier Theorem [1] (see also [4]): given two Radon measures μ_1, μ_2 on $\mathbb{R}^n$, both absolutely continuous with respect to the Lebesgue measure and such that $\mu_1(\mathbb{R}^n) = \mu_2(\mathbb{R}^n)$, there exists a convex function $\varphi : \mathbb{R}^n \to [0,\infty]$ and a Borel measurable map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(x) = \nabla\varphi(x)$ at a.e. $x \in \mathbb{R}^n$ and T pushes forward μ_1 into μ_2, i.e.

$$\int_{\mathbb{R}^n} H(y) d\mu_2(y) = \int_{\mathbb{R}^n} H(T(x)) d\mu_2(x)\,, \tag{3}$$

for every Borel function $H : \mathbb{R}^n \to [0,\infty]$. The mass transportation approach to Theorem 1 allows also to deduce a quantitative stability estimate on the maximality of $w = a1_E$, see Corollary 2 below. We pass now to prove Theorem 1.

Proof of Theorem 1. By (2), as $F(r,\cdot)$ is continuous for a.e. $r \in \mathbb{R}^+$, we deduce that $F(r,0) = 0$ for a.e. $r \in \mathbb{R}^+$. We let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, and denote by σ the $(n-1)$-dimensional Hausdorff measure restricted to S^{n-1}.

Step one: Let us fix $u \in X$ and construct an auxiliary function $v = a\,1_G$ by letting

$$G := \left\{ x \in \mathbb{R}^n : |x| < \kappa\left(\frac{x}{|x|}\right)\right\}\,,$$

where we have introduced $\kappa : S^{n-1} \to \mathbb{R}^+$,

$$\kappa(\nu) := \left(\frac{n}{a^p}\int_0^\infty u(r\nu)^p r^{n-1} dr\right)^{1/n}\,, \quad \nu \in S^{n-1}\,. \tag{4}$$

Note that $v(r\nu) = a\,1_{[0,\kappa(\nu)]}(r)$, and that the value of $\kappa(\nu)$ has been chosen so that the measures

$$1_{\mathbb{R}^+}(r) u(r\nu)^p\, r^{n-1}\, dr \quad \text{and} \quad 1_{\mathbb{R}^+}(r) v(r\nu)^p\, r^{n-1}\, dr\,,$$

have the same total mass on $\mathbb{R}$. For every $\nu \in S^{n-1}$, let T_ν denote the map given by Brenier theorem. By construction T_ν is increasing on $\mathbb{R}$, moreover, thanks to (3) we have

$$\int_{\mathbb{R}^+} H(r) v(\nu r)^p r^{n-1} dr = \int_{\mathbb{R}^+} H(T_\nu(r)) u(\nu r)^p r^{n-1} dr\,, \tag{5}$$

for every Borel function $H : \mathbb{R} \to [0,\infty]$: in particular $T_\nu(r) \in [0, k(\nu)]$ for a.e. $r \in \mathbb{R}$. Note also that, as $0 \le u \le a$, we clearly have

$$T_\nu(r) \le r\,, \quad \text{for a.e. } r \in \mathbb{R}^+. \tag{6}$$

We are going to prove that $\mathcal{F}(u) \le \mathcal{F}(v)$. By (2) we have that

$$\mathcal{F}(u) = \int d\sigma(\nu) \int_{\mathbb{R}^+} F(r, u(r\nu)) r^{n-1} dr \le \int d\sigma(\nu) \int_{\mathbb{R}^+} \frac{F(r,a)}{a^p} u(r\nu)^p r^{n-1} dr\,, \tag{7}$$

while at the same time, thanks to (5)

$$\begin{aligned}\mathcal{F}(v) &= \int d\sigma(\nu)\int_0^{\kappa(\nu)} F(r,a)r^{n-1}dr = \int d\sigma(\nu)\int_{\mathbb{R}^+}\frac{F(r,a)}{a^p}v(r\nu)^p r^{n-1}dr \\ &= \int d\sigma(\nu)\int_{\mathbb{R}^+}\frac{F(T_\nu(r),a)}{a^p}u(r\nu)^p r^{n-1}dr\,,\end{aligned}$$

By (H1) and (6) it follows immediately that $\mathcal{F}(u)\le\mathcal{F}(v)$.

Step two: We are going to prove that $\mathcal{F}(v)\le\mathcal{F}(w)$. We start by noticing that $|E|=|G|$. Indeed by (4)

$$|G| = \int\frac{\kappa(\nu)^n}{n}\,d\sigma(\nu) = \frac{1}{a^p}\int_{\mathbb{R}^n}u^p = |E|\,.$$

In particular $|E\setminus G| = |G\setminus E|$, and, without loss of generality, $|E\setminus G|>0$. Consider the Brenier map $T:\mathbb{R}^n\to\mathbb{R}^n$ between $1_{E\setminus G}(x)dx$ and $1_{G\setminus E}(y)dy$. By (3),

$$\int_{E\setminus G} H(y)dy = \int_{G\setminus E} H(T(x))dx\,, \tag{8}$$

for every Borel function $H:\mathbb{R}^n\to[0,\infty]$. On choosing $H(y)=F(y,a)$ we find

$$\int_{E\setminus G} F(|x|,a)dx = \int_{G\setminus E} F(|T(x)|,a)dx\,, \tag{9}$$

while, on taking $H(y)=1_{E\setminus G}(y)$, we prove that $T(x)\in E\setminus G$ for a.e. $x\in G\setminus E$. As E is a ball, this last remark implies that

$$|T(x)|\le|x|\,,\quad \text{for a.e. } x\in G\setminus E\,. \tag{10}$$

On combining (10) with (9) we get

$$\begin{aligned}\mathcal{F}(w) &= \int_{G\cap E}F(|x|,a)dx + \int_{E\setminus G}F(|x|,a)dx \\ &= \int_{G\cap E}F(|x|,a)dx + \int_{G\setminus E}F(|T(x)|,a)dx \\ &\ge \int_{G\cap E}F(|x|,a)dx + \int_{G\setminus E}F(|x|,a)dx = \mathcal{F}(v)\,,\end{aligned} \tag{11}$$

and the conclusion follows.

Let us now assume that for every $s\in\mathbb{R}^+$ the function $F(\cdot,a)$ is strictly decreasing, and consider a function $u\in X$ that maximizes $\mathcal{F}$ on X, i.e. such that $\mathcal{F}(u)=\mathcal{F}(w)$. We want to show that $u=w$ a.e. on $\mathbb{R}^n$. Let us prove that $G=E$ up to null sets. Indeed, let R denote the radius of the ball E. If $|G\setminus E|>0$, then we can consider T and repeat the above argument. Since $F(\cdot,a)$ is strictly decreasing and equality holds in (11), we find that $|T(x)|=|x|$ for a.e. $x\in G\setminus E$. Thus $|T(x)|\ge R$ for a.e. $x\in\mathbb{R}$; but $T(x)\in E\setminus G$ for a.e. $x\in G\setminus E$, therefore it must be $|G\setminus E|=0$, a contradiction. As $G=E$ up to null sets, we have $\kappa(\nu)=R$ for every $\nu\in S^{n-1}$. The equality sign in (7) implies that, for σ-a.e. $\nu\in S^{n-1}$, $T_\nu(r)=r$ for a.e. $r\in\{t:u(\nu t)>0\}$. As $0\le T_\nu\le\kappa(\nu)=R$, by (5) and (6) we deduce that $\{t:u(\nu t)>0\}\subset[0,R]$ for σ-a.e. $\nu\in S^{n-1}$. On applying (5) to $H=1_{\{t:u(\nu t)>0\}}$ we deduce $u(\nu r)=a$ on $\{t:u(\nu t)>0\}$, therefore that $u(\nu r)=a\,1_{[0,R]}(r)$. In particular $u=w$ a.e. on $\mathbb{R}^n$. □

We come now to a quantitative stability estimate:

Corollary 2. *Under the assumptions of Theorem 1, let us assume the existence of $\lambda > 0$ such that, whenever $0 < r_1 < r_2$,*

$$F(r_1, a) \geq F(r_2, a) + \lambda(r_2 - r_1)\,. \tag{12}$$

Then, for every $u \in X$ we have that

$$\int_{\mathbb{R}^n} |u - w|^p \leq C(n, p, a)\sqrt{\frac{\mathcal{F}(w) - \mathcal{F}(u)}{\lambda}}\,. \tag{13}$$

where $C(n, p, a)$ is a constant depending only on n, p and a.

Proof. Let $\delta := \mathcal{F}(w) - \mathcal{F}(u)$. Thanks to (12), from (7) and (11) we find that

$$\delta \;\geq\; \lambda \int_{G\setminus E} (|x| - |T(x)|)dx\,, \tag{14}$$

$$\delta \;\geq\; \lambda \int d\sigma(\nu) \int_0^\infty (r - T_\nu(r))\frac{u(r\nu)^p}{a^p} r^{n-1} dr\,. \tag{15}$$

We now consider (14) and (15) separately:

Step one: Let $\varepsilon \in (0, R)$, then $(R+\varepsilon)^n \leq R^n + c(n)R^{n-1}\varepsilon$. Thus

$$\begin{aligned}\frac{1}{a^p}\int_{\mathbb{R}^n} |w - v|^p &= |E\Delta G| = 2|G \setminus E| \leq 2\{|G \setminus B_{R+\varepsilon}| + |B_{R+\varepsilon} \setminus B_R|\}\\ &\leq C(n)\left\{|G \setminus B_{R+\varepsilon}| + \varepsilon R^{n-1}\right\}\,.\end{aligned}$$

If $x \in G \setminus B_{R+\varepsilon}$ then $|x| \geq R+\varepsilon \geq |T(x)|+\varepsilon$. By (14) we have $|G \setminus B_{R+\varepsilon}| \leq (\delta/\varepsilon\lambda)$, therefore we come to

$$\frac{1}{a^p}\int_{\mathbb{R}^n} |w - v|^p \leq C(n)\left\{\frac{\delta}{\lambda\varepsilon} + \varepsilon R^{n-1}\right\}\,.$$

We minimize over $\varepsilon \in [0, R]$ and find

$$\frac{1}{a^p}\int_{\mathbb{R}^n} |w - v|^p \leq C(n)\max\left\{\sqrt{\frac{\delta R^{n-1}}{\lambda}}, \frac{\delta}{\lambda R}\right\}\,. \tag{16}$$

Step two: We start by noticing that

$$\begin{aligned}\int_{\mathbb{R}^n} |u - v|^p &= \int_G (a-u)^p + \int_{\mathbb{R}^n\setminus G} u^p \leq \int_G (a^p - u^p) + \int_{\mathbb{R}^n\setminus G} u^p = 2\int_{\mathbb{R}^n\setminus G} u^p\\ &= 2\int \tau_2(\nu)d\sigma(\nu)\,,\end{aligned}$$

where, for every $\nu \in S^{n-1}$, we have set

$$\tau_1(\nu) := \int_0^{\kappa(\nu)} u(r\nu)^p r^{n-1} dr\,, \quad \tau_2(\nu) := \int_{\kappa(\nu)}^\infty u(r\nu)^p r^{n-1} dr\,.$$

Since $a^p\kappa(\nu)^n/n = \tau_1(\nu) + \tau_2(\nu)$, we have that

$$a^p\frac{T_\nu(r)^n}{n} \;\leq\; \tau_1(\nu) + \int_{\kappa(\nu)}^r u(t\nu)t^{n-1}dt \leq \tau_1(\nu) + a^p\frac{r^n}{n} - a^p\frac{\kappa(\nu)^n}{n}\,,$$

for every $r \geq \kappa(\nu)$, i.e.,

$$T_\nu(r) \leq \left(r^n - \frac{n\tau_2(\nu)}{a^p}\right)^{1/n}, \quad \forall r \geq \kappa(\nu)\,.$$

Then by (15) we deduce that

$$\begin{aligned}
\frac{\delta}{\lambda} &\geq \int d\sigma(\nu) \int_{\kappa(\nu)}^{\infty} \left[r - \left(r^n - \frac{n\tau_2(\nu)}{a^p}\right)^{1/n}\right] u(r\nu)^p r^{n-1} dr \\
&\geq \int d\sigma(\nu) \int_{\kappa(\nu)}^{\infty} \frac{\tau_2(\nu)}{a^p} \left(r^n - \frac{n\tau_2(\nu)}{a^p}\right)^{(1/n)-1} u(r\nu)^p r^{n-1} dr \\
&\geq \int \frac{\tau_2(\nu)^2}{a^p} \left(\kappa(\nu)^n - \frac{n\tau_2(\nu)}{a^p}\right)^{(1/n)-1} d\sigma(\nu) \\
&= \frac{c(n)}{a^{p/n}} \int \tau_2(\nu)^2 \tau_1(\nu)^{(1/n)-1} d\sigma(\nu)\,.
\end{aligned}$$

By Hölder inequality

$$\int_{\mathbb{R}^n} |u-v|^p \leq 2 \int \tau_2(\nu) d\sigma(\nu) \leq C(n) \sqrt{a^{p/n} \frac{\delta}{\lambda}} \sqrt{\int \tau_1(\nu)^{1-(1/n)} d\sigma(\nu)}\,.$$

By Jensen inequality for concave functions,

$$\begin{aligned}
\int \tau_1(\nu)^{1-(1/n)} d\sigma(\nu) &\leq C(n) \left(\int \tau_1(\nu) d\sigma(\nu)\right)^{1-(1/n)} \\
&\leq C(n) \left(\int a^p \frac{\kappa(\nu)^n}{n} d\sigma(\nu)\right)^{1-(1/n)} = C(n)\,,
\end{aligned}$$

and we come to conclude that

$$\int_{\mathbb{R}^n} |u-v|^p \leq C(n) \sqrt{a^{p/n} \frac{\delta}{\lambda}} \tag{17}$$

Step three: As $\int |w-u|^p \leq 2^p$, (13) follows trivially whenever $\delta \geq \leq$. Let us now assume that $\delta \leq \lambda$, then (13) is easily deduced from (16) and (17). $\square$

References

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